

3D transformation for point rotating reference frames and localized neutrinos in a rotating electromagnetic field

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The problem of Dirac's equation in a rotating electromagnetic field can be reduced to the stationary by using a transformation for point rotating reference frames. The general form of the non-Galilean transformation is deduced in the paper. For the non-Galilean transformation time is different in the rotating and resting frame. This transformation contains a constant, with the dimension of time. The constant is assumed to be fundamental.

The transformation forms the necessary condition for the existence of periodic, bounded and square integrable solutions in both the rotating and resting frame. Variety of such solutions exists. Among the solutions massless states are identified. They describe localized neutrinos with invariable spin. The states in the resting frame are not stationary but they have a good chance to be stable.

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INTRODUCTION

Solutions of the Dirac equation in a rotating electromagnetic field relate to the non-stationary problem. The problem may turn into the stationary by a transition to a rotating reference frame. Solutions in this frame correspond to stationary states (Landau levels [1]).

The transition is realized with help of a transformation of the wave function and coordinates. The transformation of the wave function is uniquely determined by the electromagnetic potential, but for coordinates as the Galilean as non-Galilean transformation may be used. Two last terms correspond to the same and different time in the rotating and resting frame, respectively.

Two types of rotation exist in nature. The first with one axis of rotation pertains to mechanics. Time in such a rotating frame depends on the frequency and distance from the axis of rotation.

The second type with the rotational axis at each point may be associated with any rotating field. A reference frame (point rotating reference frame) may be connected with the field. Such a frame (optical indicatrix or index ellipsoid) is used in crystallooptics more 100 years. The concept of the frame can be applicable not only for optics and quantum mechanics, but also and in the general relativity [2].

The paper reviews the second type. Coordinates of 2D transformation are angle and time of the cylindrical coordinate system. For 3D case the coordinate along the axis of rotation is added, while the cylindrical radius remains unchanged. Such a frame is free of centrifugal forces.

It is well known that time in two frames, moving with different velocities, is different. But is it valid for two frames rotating with different frequency? From the viewpoint of modern physics the positive answer to this question seems more preferable. Moreover the non-Galilean transformation must contain a fundamental physical con-

stant with the dimension of time. This constant is lacking on the list of basis physical constants like the speed of light, Planck's constant, electron charge, and so on. Possibly, some problems of contemporary physics are connected with this constant. This defines the significance of this issue.

Some theoretical and experimental aspects of non-Galilean transformation in application to optics are presented in [2]. A simple 2D transformation was discussed in this paper as an example. Consideration has been performed to evaluate the time difference in the rotating frame relative to the resting one by using existing experimental data in optics [3].

THE DIRAC EQUATION

We consider Dirac's equation

$$i\hbar \frac{\partial}{\partial t} \Psi = c\alpha(\mathbf{p} - \frac{e}{c}\mathbf{A})\Psi + \beta mc^2 \Psi = 0 \quad (1)$$

in an electromagnetic field, consisting of a plane traveling circularly polarized electromagnetic wave and a constant magnetic field, with the potential

$$A_1 = -\frac{1}{2}H_3y + \frac{1}{k}H \cos(\epsilon\Omega t - kz), \quad (2)$$

$$A_2 = \frac{1}{2}H_3x + \frac{1}{k}H \sin(\epsilon\Omega t - kz), \quad (3)$$

where Ω is the frequency, $\epsilon = +1$ and $\epsilon = -1$ corresponds to right and left-hand polarization, respectively, $k = \epsilon\Omega/c$ is the propagation constant, $\epsilon = +1$ and $\epsilon = -1$ are used when the wave propagates along the z -axis and the opposite direction, insertion of ϵ, ϵ is justified because the system is not symmetric when the signs of ϵ, ϵ are changed independently, Ω always remains positive, c is the speed of light, α_k, β are Dirac's matrices, H_3 is the constant magnetic field, H is the amplitude of

the magnetic field of the wave. It is well known that the amplitude of the electric and magnetic fields is the same for the given wave. H is used for comparison with the constant magnetic field H_3 .

The transformation of the wave function

$$\tilde{\Psi} = \exp \frac{1}{2} \alpha_1 \alpha_2 (\epsilon \Omega t - \epsilon \frac{\Omega}{c} z) \Psi, \quad (4)$$

reduces Eq. (1) to the stationary form. The transformation should be accompanied by the transformation of coordinates. In a sense the problem is similar to the problem in mechanics. For transition to the reference frame of center mass the Lorentz transformation should be used. The reverse transition in the laboratory frame can be performed after studies of temporal and spatial behavior of a object.

THE TRANSFORMATION OF COORDINATES

The transformation of coordinates connects the cylindrical angle φ , coordinate along the axis of rotation z and time t in the resting and rotating frames. The cylindrical radius r remains invariable. The dependence of the angle is defined by the transformation of spinor $\tilde{\varphi} = \varphi - \epsilon \Omega t + \epsilon \Omega z/c$. The dependence of \tilde{z}, \tilde{t} on z, t is taken to be a linear form of coordinates with coefficients depending on Ω .

The 3D transformation is determined by following general assumptions:

- the dependence of \tilde{z}, \tilde{t} from z, t has Lorentz's shape

$$\tilde{z} = a\varphi + \frac{z + vct}{\sqrt{1 - v^2}}, \quad \tilde{t} = b\varphi + \frac{zv/c + t}{\sqrt{1 - v^2}}, \quad (5)$$

where a, b, v are parameters depending on Ω .

- the principle of the velocity of light constancy is kept: if the velocity in the resting frame $V \equiv z/t = \epsilon_v c$ then in the rotating frame $\tilde{V} \equiv \tilde{z}/\tilde{t} = \epsilon_v c$, where $\epsilon_v^2 = 1$. This condition in the general case would give two relations between coefficients. However one relation is fulfilled automatically, because of Lorentz's shape (5), and only one remains

$$\epsilon_v cb = a.$$

- the analogous principle of some frequency constancy exists: if $\omega \equiv \varphi/t = \epsilon_\omega \tau$ then $\tilde{\omega} \equiv \tilde{\varphi}/\tilde{t} = \epsilon_\omega \tau$, where $\epsilon_\omega^2 = 1$ and τ is a positive constant with the dimension of time. The condition should be valid for arbitrary V . From this two relations follows

$$v = \frac{\epsilon_\omega \epsilon \tau \Omega}{\sqrt{1 + \tau^2 \Omega^2}}, \quad |v| < 1, \quad (6)$$

$$b = \epsilon_\omega \tau [\epsilon_\omega \epsilon \tau \Omega - (1 - \sqrt{1 + \tau^2 \Omega^2})]. \quad (7)$$

The rotating frame gains the velocity vc along the z -axis.

Finally the transformation has the form

$$\begin{aligned} \tilde{\varphi} &= \varphi - \epsilon \Omega t + \epsilon \Omega z/c, \\ \tilde{z} &= -\epsilon_v cb \varphi + \sqrt{1 + \tau^2 \Omega^2} z + \epsilon_\omega \epsilon \tau \Omega ct, \\ \tilde{t} &= -b \varphi + \epsilon_\omega \epsilon \frac{\tau \Omega}{c} z + \sqrt{1 + \tau^2 \Omega^2} t. \end{aligned} \quad (8)$$

An attempt to evaluate τ is made [2] on the basis of the J. P. Campbell, and W. H. Steier experiment [3]. Using the 2D non-Galilean transformation $\tilde{\varphi} = \varphi - \Omega t$, $\tilde{t} = -b\varphi + t$, where b is a parameter depending on Ω , and expanding b in power series $b = \tau_0 + \tau^2 \Omega + \tau_2^3 \Omega^2 + \dots$, it was shown that if $\tau_0 \neq 0$ the upper boundary for $\tau_0 \sim 10^{-23}$ sec. If τ_0 is exactly equals zero the upper boundary for τ , which can be measured optically, is $\sim 10^{-17} - 10^{-18}$ sec. Even for frequencies of the order of $100 GHz$ the product $\tau \Omega < 10^{-6}$. The velocity $v \sim 100 m/sec$ for this value of $\tau \Omega$.

Below we operate with the upper boundary of $\tau \sim 10^{-17}$.

SOLUTIONS

We are looking for solutions of the Dirac equation. The solutions should be periodic, bounded and square integrable perpendicularly to the z -axis. In the rotating frame they are stationary, whereas in the resting, non-stationary. Moreover in the paper only exact solutions are considered.

Desirable solutions in the rotating frame have following form

$$\tilde{\Psi} = \exp(-\frac{i\tilde{E}}{\hbar}\tilde{t} + \frac{i\tilde{p}}{\hbar}\tilde{z} - in\tilde{\varphi} + D)\psi, \quad (9)$$

were \tilde{E} and \tilde{p} is the "energy" and "momentum" along the z -axis, n is an integer, $D = -d\tilde{r}^2/2 + d_1\tilde{x} + d_2\tilde{y}$, $\tilde{x} = r \cos \tilde{\varphi}$, $\tilde{y} = r \sin \tilde{\varphi}$. The wave function ψ in the rotating frame obeys the equation

$$\begin{aligned} &\{-E + i\hbar\Omega(\epsilon - \alpha_3\epsilon)[\tilde{x}d_2 - \tilde{y}d_1 + \frac{1}{2}\alpha_1\alpha_2] - \\ &-i\hbar c\alpha_1(-\tilde{x}d + d_1) - i\hbar c\alpha_2(-\tilde{y}d + d_2) + \alpha_3 pc - \\ &-(\alpha_2\tilde{x} - \alpha_1\tilde{y})e\frac{1}{2}H_3 - \alpha_1 e\frac{1}{k}H + \beta mc^2\}\psi = 0, \end{aligned} \quad (10)$$

where parameters E and p are defined as

$$E = \sqrt{1 + \tau^2 \Omega^2} \tilde{E} - \epsilon_\omega \epsilon \tau \Omega \tilde{p} c - \epsilon n \hbar \Omega, \quad (11)$$

$$pc = \sqrt{1 + \tau^2 \Omega^2} \tilde{p} c - \epsilon_\omega \epsilon \tau \Omega \tilde{E} - \epsilon n \hbar \Omega. \quad (12)$$

The condition

A necessary condition for existence of periodic and bounded solutions is

$$\tau[\epsilon_\omega(\sqrt{1 + \tau^2 \Omega^2} - 1) + \epsilon \tau \Omega](\tilde{E} - \tilde{p} c \epsilon_v) = \hbar n, \quad (13)$$

n must be an integer, because $\tilde{\varphi}$ is normalized so that the variation of $\varphi \rightarrow \varphi + 2\pi$ results the variation $\tilde{\varphi} \rightarrow \tilde{\varphi} + 2\pi$ (8).

The reverse transition to the resting frame is realized by the transformation coordinates and the wave function with the tilde to that without the tilde.

$$-i\frac{\tilde{E}\tilde{t}}{\hbar} + i\frac{\tilde{p}\tilde{z}}{\hbar} - in\tilde{\varphi} = -i\frac{Et}{\hbar} + i\frac{pz}{\hbar}$$

$$\tilde{x} = x \cos(\epsilon\Omega t - kz) + y \sin(\epsilon\Omega t - kz), \quad (14)$$

$$\tilde{y} = y \cos(\epsilon\Omega t - kz) - x \sin(\epsilon\Omega t - kz), \quad (15)$$

Because of the expressions for \tilde{x}, \tilde{y} states in resting frame are not stationary.

Eq. (10) with definitions (11), (12) coincides with the equation obtained by means of the Galilean transformation $\tilde{\varphi} = \varphi - \epsilon\Omega t + \epsilon\Omega z/c$, $\tilde{z} = z$, $\tilde{t} = t$. The constant τ and $in\tilde{\varphi}$ disappears from Eq. (10). The only difference is the condition (13).

Note that in a sense the optical indicatrix (index ellipsoid) of threefold electrooptical crystal, which used in [3], possesses some properties of the two-component spinor. The rotation of the electric field on an angle produces rotation of the optical indicatrix on a half of this angle. However, there is a principal dissimilarity of the behavior the circularly polarized light wave in the single-sideband modulator, under the action of a rotating (modulating) electric field, and spinor in the field of rotating electromagnetic wave. This dissimilarity is the polarization reversal at the modulator output. In this case the constant τ is not vanished.

The even part $[b(\Omega) + b(-\Omega)]/2$ gives an asymmetry in the frequency shift whereas the odd $[b(\Omega) - b(-\Omega)]/2$ gives the same shift for right- and left-hand rotations of the circularly polarized light wave. In the transformation (8) b has as the even $-(1 - \sqrt{1 + \tau^2\Omega^2})$ as odd part $\epsilon\omega\epsilon\tau\Omega$, the constant $\tau_0 = 0$.

Point rotating reference frames are free from centrifugal forces. Under the condition of (13) desirable solutions are possible in both the rotating and resting frame. This fact is a good indicator of the solution stability.

Solutions can be classified by the form of spinor ψ in (10). A constant spinor describes the ground state. A polynomial in \tilde{x}, \tilde{y} corresponds to excited states.

Ground state

Spinor ψ of the ground state in normalized units is $\psi = N\psi_0 \exp \Phi$,

$$\psi_0 = \begin{pmatrix} h\mathcal{E} \\ -\epsilon(\mathcal{E} + \epsilon)(\mathcal{E} - \mathcal{E}_0) \\ \epsilon\epsilon h\mathcal{E} \\ -\epsilon(\mathcal{E} - \epsilon)(\mathcal{E} - \mathcal{E}_0) \end{pmatrix}, \quad (16)$$

where the normalized constant is

$$N = \frac{\sqrt{d} \exp(-d_2^2/2d)}{\sqrt{2\pi} \sqrt{h^2\mathcal{E}^2 + (\mathcal{E}^2 + 1)(\mathcal{E} - \mathcal{E}_0)^2}}. \quad (17)$$

The "normalized energy" \mathcal{E} obeys the characteristic equation

$$\mathcal{E}(\mathcal{E} + \Lambda) - 1 - \frac{\mathcal{E}h^2}{\mathcal{E} - \mathcal{E}_0} = 0, \quad \Lambda = \frac{2\epsilon\tilde{p}c - \hbar\Omega}{mc^2}, \quad (18)$$

in normalized units

$$\mathcal{E} \equiv \frac{E\epsilon - pc\epsilon}{mc^2}, \quad \mathcal{E}_0 = \frac{2d\hbar}{\Omega m}, \quad h = \frac{e}{kmc^2}H. \quad (19)$$

Parameters Φ, d, d_1, d_2 are defined as follows

$$\Phi = -\frac{i\tilde{E}}{\hbar}\tilde{t} + \frac{i\tilde{p}}{\hbar}\tilde{z} - in\tilde{\varphi} - \frac{1}{2}\alpha_1\alpha_2(\epsilon\Omega t - \epsilon\frac{\Omega}{c}z) - \frac{1}{2}d\tilde{r}^2 + d_1\tilde{x} + d_2\tilde{y}, \quad (20)$$

$$d = \pm \frac{eH_3}{2\hbar c} > 0, \quad d_1 = -id_2, \quad d_2 = \frac{cdh}{\Omega(\mathcal{E} - \mathcal{E}_0)}. \quad (21)$$

The desirable solutions are localized in the cross section with the size of the order l_d

$$l_d \sim \frac{1}{\sqrt{d}} = \sqrt{\frac{2\hbar c}{eH_3}}. \quad (22)$$

For definiteness we consider $d = -eH_3/2\hbar c$. For the $d = +eH_3/2\hbar c > 0$, the wave function is defined as $\psi_+ = \epsilon\alpha_1\alpha_3\beta\psi_0$ with the simultaneous sign change of \mathcal{E}_0 .

There exists variety of excited states, in particular,

$$\psi_e = N_{e1}\psi_0(1 - i\frac{d}{d_2}\tilde{x} - \frac{d}{d_2}\tilde{y}), \quad (23)$$

where ψ_0 as well as parameters d, d_1, d_2 are the same as in (16) and (21), whereas the parameter Λ differs $\Lambda = (2\epsilon\tilde{p}c - 3\hbar\Omega)/mc^2$.

Obviously, wave functions (16), (23) cannot be presented as a small and large two-component spinor. It means that the difference $E^2 - m^2c^2$ cannot be small and these solutions correspond only to the relativistic case.

Consider the condition (13) in the resting frame. We obtain, with help of (11), (12) at $\tau\Omega \ll 1$ in the first approximation

$$\epsilon(E - pc\epsilon_v) \approx \frac{\hbar n}{\tau^2\Omega}. \quad (24)$$

For the discussed above value of $\tau \sim 10^{-17}$, $\tau\Omega \sim 10^{-6}$ the right part is of the order of $10^3 m_p c^2$, where m_p is of the order of the proton mass. For electron it corresponds huge energy. For heavier particles this is also in line with the high energy physics, especially, if the real value of the constant τ is less than 10^{-17} .

Condition (11), (12), (13) and the characteristic equation allows to determine parameters E and pc and, consequently, the wave function in the resting frame for $n \neq 0$. Analogously E and pc can be calculated for $n = 0$ at $\varepsilon_v = -\varepsilon\varepsilon..$

In this paper we don't consider the case $n \neq 0$ with too cumbersome formulas and restrict ourselves to the interesting case $n = 0$, $\varepsilon_v = \varepsilon\varepsilon$.

Localized neutrinos

Consider $\varepsilon_v = \varepsilon\varepsilon$. The equality $\tilde{E} = \tilde{p}c\varepsilon_v$ results in the equality $\varepsilon E = \varepsilon pc$ with help of Eqs. (11), (12). In this case the characteristic equation is fulfilled only for $m = 0$. The massless ground state in these conditions is described by the wave function

$$\Psi = \left(-\frac{d_2^2}{2d}\right) \sqrt{\frac{d}{2\pi}} \begin{pmatrix} 0 \\ \varepsilon\varepsilon \\ 0 \\ -1 \end{pmatrix} \exp \Phi, \quad (25)$$

$$\Phi = -i\frac{Et}{\hbar} + i\frac{pz}{\hbar} - \frac{1}{2}\alpha_1\alpha_2(\varepsilon\Omega t - \varepsilon\frac{\Omega}{c}z) - \frac{1}{2}dr^2 + d_1\tilde{x} + d_2\tilde{y}, \quad (26)$$

$$d_2 = -\frac{\varepsilon e}{2\hbar\Omega}H. \quad (27)$$

The existence of this exact solution is easy to verify by substituting (25) into the initial equation (1).

Excited states also exist, in particular, the solution obtained from Eq. (23).

Because solutions are not stationary it is relevant to consider the average values of operators of energy E_a , momentum \mathbf{p}_a and spin. For the given case of the localization perpendicular to the z -axis the average value of an operator P is defined by the integral over all cross-section

$$P_a = \int \Psi^* P \Psi dx dy.$$

The integral can be calculated exactly for all average values below.

The average values of E_a and p_{za} obey the same relation as the parameters E and p

$$\varepsilon E_a = \varepsilon E - \hbar\Omega + \frac{ceH_3}{2\Omega} \left(\frac{H}{H_3}\right)^2 = \varepsilon p_{za}c, \quad (28)$$

The values of p_{xa}, p_{ya} change with t, z similarly to potential A_1, A_2

$$p_{xa} = \frac{\varepsilon e}{2\Omega}H \cos(\varepsilon\Omega t - \varepsilon\frac{\Omega}{c}z), \quad (29)$$

$$p_{ya} = \frac{\varepsilon e}{2\Omega}H \sin(\varepsilon\Omega t - \varepsilon\frac{\Omega}{c}z). \quad (30)$$

The average components of spin are invariable

$$s_3 = \frac{\hbar}{2}, \quad s_1 = s_2 = 0. \quad (31)$$

Other interesting massless solution follows from the previous by $E = p = 0$. The wave function, average energy and the component of momentum p_{za} depends only on parameters of the electromagnetic field.

CONCLUSION

The 3D non-Galilean transformation for point rotating reference frames is uniquely determined by the following general assumptions:

- Lorentz's form of the dependence of the "Cartesian coordinates" in the rotating and resting frame;
- The principle of the velocity of light constancy;
- Similar principle of some frequency constancy.

Surprisingly the frequency, in contrast to the velocity, is not the limit frequency in this transformation. A consequence of this transformation is the condition of the existence of periodic, bounded and localized solutions in the rotating and resting frame. The non-Galilean transformation contains a constant with the dimension of time. This fundamental constant defines processes in the very small intervals of time and length. Therefore the transformation is of particular interest to the standards of time and length and also for nanotechnology.

Periodic, bounded and square integrable solutions of Dirac's equation in the rotating electromagnetic field have been considered. They describe only relativistic fermions. These fermions are localized in the small cross-section with the size of the order of l_d . All states that have been found in this paper are exact solutions. Among the solutions, massless solutions exist. These solutions are described non-stationary (but, possibly, stable) localized neutrinos with invariable spin. Apparently, such solutions can be used in quantum field theory and high-energy physics.

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